Non-Fermi liquids and the Wiedemann-Franz law

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Abstract

A general discussion of the ratio of thermal and electrical conductivities in non-Fermi liquid metals is given. In metals with sharp Drude peaks, the relevant physics is correctly organized around the slow relaxation of almost-conserved momenta. While in Fermi liquids both currents and momenta relax slowly, due to the weakness of interactions among low energy excitations, in strongly interacting non-Fermi liquids typically only momenta relax slowly. It follows that the conductivities of such non-Fermi liquids are obtained within a fundamentally different kinematics to Fermi liquids. Among these strongly interacting non-Fermi liquids we distinguish cases with only one almost-conserved momentum, which we term hydrodynamic metals, and with many patchwise almost-conserved momenta. For all these cases, we obtain universal expressions for the ratio of conductivities that violate the Wiedemann-Franz law. We further discuss the case in which long-lived 'cold' quasiparticles, in general with unconventional scattering rates, coexist with strongly interacting hot spots, lines or bands. For these cases, we characterize circumstances under which non-Fermi liquid transport, in particular a linear in temperature resistivity, is and is not compatible with the Wiedemann-Franz law. We suggest the likely outcome of future transport experiments on $CeCoIn_5$, $YbRh_2Si_2$ and $Sr_3Ru_2O_7$ at their critical magnetic fields.

1 The Wiedemann-Franz law

The Wiedemann-Franz law for the Lorenz ratio of thermal conductivity κ to electrical conductivity σ, at low temperature T,

$$L \equiv \frac{\kappa}{\sigma T} = \frac{\pi^2}{3} \equiv L_0 \,, \tag{1}$$

in units with $k_B = e = 1$, is a robust feature of Fermi liquids at low temperature [1]. In a non-Fermi liquid metal, there is ample reason to expect the Wiedemann-Franz law to be violated. There may be additional gapless neutral collective degrees of freedom present that transport heat but not charge. There may furthermore be inelastic scattering between charged and neutral degrees of freedom that affects heat and charge transport differently. It may therefore seem surprising that in several systems exhibiting supposedly hallmark non-Fermi liquid phenomenology, such as a linear in temperature resistivity, the Wiedemann-Franz law is observed to hold at the lowest temperatures [2, 3, 4, 5, 6], while the interpretation of recently reported violations of the law in such materials has proved contentious [7, 8, 9]. On the other hand, violations of the Wiedemann-Franz law in metallic regimes at low temperatures have been reported in the underdoped cuprates [10, 11, 12], upon suppressing superconductivity, and also in the c-axis conductivity of a heavy fermion non-Fermi liquid [13]. This panoply of results clearly presents a theoretical challenge.

The thermal and electrical conductivities are subtle observables because they depend crucially on momentum relaxation in order to be finite. The interactions between charge carriers and gapless neutral degrees of freedom causing non-Fermi liquid behavior may not in themselves cause momentum relaxation. To understand the ratio of conductivities, one must therefore characterize the interplay between momentum-conserving and momentum-non-conserving interactions. If a quasiparticle description is apposite, then this question can be addressed in a careful treatment of the Boltzmann equation, e.g. [1, 14]. In this paper we shall make some simple observations about the ratio of thermal to electric conductivity in generic non-Fermi liquids. Important differences will be found between cases in which weakly interacting quasiparticles are present and those in which they are not. The tool we shall use to organize our discussion, which emphasizes the role of momentum relaxation, is the memory matrix formalism [15].

The memory matrix formalism allows a clear-headed discussion of conductivities without recourse to a quasiparticle worldview. As such it has proved useful in studying the transport of one dimensional interacting electrons [16, 17] and higher dimensional systems at quantum criticality [18, 19, 20]. The insight exploited by the memory matrix approach,

applied to the question at hand, is the following: Even if the system is strongly interacting, in a metallic phase the effects of momentum relaxation (due to quenched impurities or a lattice) on low energy processes can often be understood as a perturbation about an effective translation-invariant low energy theory. In many cases this will be synonymous with the system exhibiting a well-defined Drude peak in the optical conductivity at the low temperatures of interest. Note that a Drude peak certainly does not imply a quasiparticle description of transport. The memory matrix method then proceeds to isolate precisely the quantities that may be treated perturbatively in these cases.

The memory *matrix* formalism is to be distinguished from the earlier 'memory *function* approximation' developed in e.g. [21, 22, 23]. In that approach one works perturbatively in the relaxation of the electrical current rather than the momentum. This is most appropriate for non-interacting systems in which the total current $\vec{J}$ itself is a conserved operator (i.e. $\dot{\vec{J}} = 0$) prior to impurity and other scatterings. For the strongly interacting non-Fermi liquids we wish to consider below, the total current itself is not an almost conserved operator and one should instead focus on momentum conservation. The momentum and current operators are typically not equal in these systems. The memory matrix furthermore includes various projection operators, see for instance equation (7) below, that render it an exact expression rather than an approximation.

In this paper, for the most part, we avoid detailed computations in specific theories and focus instead on a rough classification of the different possible kinematic regimes that can control transport in the theories of interest. Our results therefore take the form of relationships between different physical quantities that pertain depending on the kinematics of the system. For instance, part of the power of the Wiedemann-Franz law (1) is that the right hand side is a pure number that is computable using Fermi liquid theory. Effects sensitive to processes external to Fermi liquid theory, in particular the quasiparticle lifetime, have cancelled out of the final expression. We shall give a memory matrix description of this fact below, and, from our more general perspective, establish conditions under which an analogous universality may be achieved for non-Fermi liquids.

Our considerations apply to the following cases:

1. *Non-Fermi liquids with no quasiparticles.* Section 4. These may include two dimensional nematic quantum critical points and Fermi surfaces coupled to emergent gauge fields, in which the full Fermi surface goes critical. The kinematics of almost-conserved quantities is entirely different to Fermi liquid theory and the Wiedemann-Franz law is expected to not hold even approximately. Currents are not approximately conserved.

The only almost-conserved quantities are momenta, of which there may be one or many (if the patches of a strongly interacting Fermi surface decouple). For the case of only one conserved momentum, we refer to these as 'hydroynamic metals', we obtain the ratios of conductivities

$$\frac{\overline{\kappa}}{\sigma T} = \frac{1}{T^2}\frac{\chi_{QP}^2}{\chi_{JP}^2} \qquad \text{and} \qquad \frac{\kappa}{\sigma T} \ll 1\,. \tag{2}$$

Here $\overline{\kappa}$ is the thermal conductivity at zero electric field, while κ is the conductivity at zero electric current. The χ_{AB} are static susceptibilities involving the operators for the total momentum $\vec{P}$, electric current $\vec{J}$ and heat current $\vec{Q}$. While the specific values and temperature dependence of these susceptibilities will depend on details of the theory at all energy scales, the left hand result in (2) is universal in the sense that the mechanism of momentum relaxation has once again cancelled from the ratio of conductivities. This universality is analogous to the Wiedemann-Franz law. Cases with many patchwise conserved momenta are discussed in section 4.2, where we obtain

$$\frac{\overline{\kappa}}{\sigma T} \sim \frac{\kappa}{\sigma T} \sim \frac{1}{T^2}\left\langle \frac{\chi_{PQ}^2}{\chi_{PJ}^2} \right\rangle . \tag{3}$$

While $\kappa = \overline{\kappa}$ in a Fermi liquid, $\kappa \sim \overline{\kappa}$ showing the same temperature scaling without equality is a diagnostic for strongly interacting transport with patchwise conserved momenta. The angled brackets refer to a specific average over the Fermi surface. This expression also exhibits a universality in that the temperature dependence is independent of mechanism of momentum relaxation. It is weaker than (2) because the coefficient of the temperature dependence is sensitive to the geometry of the Fermi surface.

2. *Non-Fermi liquids with long-lived quasiparticles.* Section 5. These refer to systems in which some or all of the charge and heat carriers remain long lived, despite the system exhibiting unconventional metallic transport. These may include finite wavevector quantum critical points, such as spin and charge density wave transitions in metals, which have hot and cold fermionic excitations. They may also include situations where one electron band goes critical but others do not. Finally these include cases in which the whole Fermi surface is critical but the fermions retain a quasiparticle character. The Wiedemann-Franz law will be obeyed in these systems at low temperatures if the long-lived electronic quasiparticles dominate the charge and heat transport at low temperatures and scatter elastically. We discuss circumstances under which this may be expected to occur. Depending on whether the electronic quasiparticles have a

conventional or unconventional scattering rate, the Wiedemann-Franz law will either emerge together with a crossover to Fermi liquid transport at low temperatures, or will coexist with non-Fermi liquid transport. We highlight a scenario in which the Wiedemann-Franz law can coexist at low temperatures with a (widely observed) linear in temperature electrical resistivity: if the 'cold' degrees of freedom scatter elastically off the 'hot' modes, which in turn relax momentum more efficiently than the cold excitations. In these circumstances, the hot modes act as 'generalized phonons'.

The above outline makes clear that the status of the Wiedemann-Franz law at low temperatures in a non-Fermi liquid provides immediate insight into the nature of the excitations in the system. In section 6 we use the simple observations summarized above to organize the existing experimental data on the Wiedemann-Franz law in heavy fermions, ruthenates and the cuprates. See table 1 below. The law has not yet been studied in certain natural temperature regimes, or ambiguous results have been obtained, allowing us to predict the outcome of future measurements.

2 The memory matrix

Let us briefly introduce the memory matrix. To follow the present paper, only the logical structure of equation (6) below need be understood. We will avoid actual evaluation of the memory matrix. The starting point are the correlators $\chi(z)$ and the static susceptibility χ

$$\chi_{AB}(z) = i\int_0^\infty dt e^{izt}\langle [A(t), B(0)]\rangle\,, \qquad \chi_{AB} = \lim_{z\to i0^+}\chi_{AB}(z)\,, \tag{4}$$

with the operators A, B being taken from the set of (i) the operators whose two point function we wish to compute, these are the total electric and thermal currents $\{\vec{J}, \vec{Q}\}$, and (ii) any almost-conserved operators that have an overlap with the currents of interest, for instance the total momentum $\vec{P}$. The quantity of interest to us is the matrix of conductivities

$$\sigma(\omega) = \frac{\chi(\omega) - \chi}{i\omega}\,. \tag{5}$$

In this formula and in all the formulae below, it is understood that the limit of real frequencies is taken from the upper half plane, $z \to \omega + i0^+$, as this is where the integrals in (4) converge. The memory matrix formalism expresses the matrix of conductivities as

$$\sigma(\omega) = \frac{1}{-i\omega + M(\omega)\chi^{-1}}\chi\,, \tag{6}$$

with the memory matrix being given by [15]

$$M_{AB}(\omega) = \int_0^{1/T} d\lambda \left\langle \dot{A}(0) \mathcal{Q} \frac{i}{\omega - \mathcal{Q}L\mathcal{Q}} \mathcal{Q}\dot{B}(i\lambda) \right\rangle . \tag{7}$$

Here L is the Liouville operator $L = [H, \cdot]$ and $\mathcal{Q}$ projects onto the space of operators orthogonal to $\{\vec{P}, \vec{J}, \vec{Q}\}$. The reason it is useful to introduce the memory matrix is that if the effects of momentum or current non-conservation are small at low energies, this quantity can be treated perturbatively because the $\dot{A}$ and/or $\dot{B}$ appearing in (7) will be small. The d.c. conductivities are

$$\sigma^{\text{d.c.}} = \lim_{\omega \to 0} \chi M(\omega)^{-1} \chi \equiv \Gamma^{-1} \chi \, . \tag{8}$$

Here we introduced the matrix of relaxation rates Γ. While χ and M will be symmetric, Γ need not be. Equation (8) expresses the d.c. conductivities as a combination of 'fast' processes described by χ, in which currents and momenta source each other, and 'slow' processes described by M. The inverse memory matrix M^{-1} will be dominated by the decay of any almost-conserved quantities in the system that overlap with the currents. The quantities of interest are then the electrical, heat and thermoelectric conductivities

$$\sigma \equiv \sigma^{\text{d.c.}}_{JJ} \, , \qquad \overline{\kappa} \equiv \frac{\sigma^{\text{d.c.}}_{QQ}}{T} \, , \qquad \alpha \equiv \frac{\sigma^{\text{d.c.}}_{JQ}}{T} \, . \tag{9}$$

Equation (6) shows that when some of the relaxation rates can be treated pertubatively – when the memory matrix approach is useful – then there will be a sharp Drude peak in the frequency dependent conductivity. We will see below that when the low energy theory has a patchwise structure in momentum space there can be many almost conserved quantities, leading in general to multiple small relaxation rates. In these cases the analytic structure of the sharp Drude peak will be more complicated than a single pole in frequency space.

3 Fermi liquids

As a first application of the memory matrix formalism, we can give a re-derivation of the Wiedemann-Franz law (1). The effective low energy and momentum-conserving theory in this case is the patchwise description of the excitations of a Fermi surface [24, 25]. Let us label the patches by θ and the momentum locally perpendicular to the Fermi surface as k. Fermi liquids have the rather special property that to lowest order in energies the different patch theories are decoupled and furthermore free. This last statement requires qualification in two dimensions due to the logarithmic growth in the strength of scattering by disorder and by soft collective particle-hole excitations at low temperatures. We return

to this point at the end of the section. Decoupled free patches leads to the fact that the quasiparticle densities $\delta n_{\theta k} = c^\dagger_{\theta k} c_{\theta k}$ are all independently conserved. The electron-electron interactions that can relax the density within patches, such as normal forward scattering and umklapp processes, are negligible at the lowest temperatures and energy scales [1]. The leading density (and, shortly, momentum) relaxing interaction is elastic scattering by quenched impurities. Thus we write

$$\frac{d}{dt}\delta n_{\theta k} = -\int d\theta' dk' \, \Gamma_{\theta\theta' k k'} \delta n_{\theta' k'} \, . \tag{10}$$

An explicit expression can be obtained for the relaxation rates per unit perpendicular momentum $\Gamma_{\theta\theta' k k'}$ from e.g. the appropriate Boltzmann equation [1]. To implement the memory matrix method, we need to assume that the scattering rate Γ is small in units of the Fermi energy μ. The Wiedemann-Franz law can in fact hold without this assumption [26]. The large number of almost conserved quantities $\delta n_{\theta k}$ means that the memory matrix method is applied in a particular way for Fermi liquids.

The almost conserved quantities of direct interest to us, the momentum and heat and electrical currents, may then be constructed patchwise out of the almost conserved densities. To lowest order in energies

$$\vec{P}_\theta = \vec{k}_{F\,\theta} \int dk \, \delta n_{\theta k} \, , \quad \vec{Q}_\theta = \vec{v}_{F\,\theta} \int dk \, \varepsilon_{\theta k} \, \delta n_{\theta k} \, , \quad \vec{J}_\theta = \vec{v}_{F\,\theta} \int dk \, \delta n_{\theta k} \, . \tag{11}$$

The quasiparticle energy $\varepsilon_{\theta k}$ vanishes at the Fermi surface and so must be kept inside the k integral. The total momenta and currents are obtained by integrating these expressions over the whole Fermi surface.

For this Fermi liquid case, we see that generically there is no hierarchy between the rates at which patchwise and total almost-conserved quantities are relaxed, as all of these quantities are tied to the relaxation of the quasiparticle density. Therefore, the memory matrix must be extended to allow the operators to run over all the patches. Because the conserved quantities are all constructed from the densities (10), it is sufficient to consider the memory matrix for these quantities. We can then write, picking some spatial direction $\vec{n}$ and setting $v_{F\,\theta} = \vec{v}_{F\,\theta} \cdot \vec{n}$,

$$\begin{aligned} \overline{\kappa} &= \frac{1}{T} \int d\theta d\theta' dk dk' \, v_{F\,\theta} \, v_{F\,\theta'} \, \varepsilon_{\theta k} \, \varepsilon_{\theta' k'} \, \sigma^{\text{d.c.}}_{\delta n_{\theta k} \delta n_{\theta' k'}} \\ &= \frac{1}{T} \int d\theta d\theta' v_{F\,\theta} \, v_{F\,\theta'} \int dk dk' \left(\Gamma^{-1}\right)_{\theta\theta' k k'} \varepsilon_{\theta k} \, \varepsilon_{\theta' k'} \, \chi_{\delta n_{\theta' k'} \delta n_{\theta' k'}} \, . \end{aligned} \tag{12}$$

In the first line $\sigma^{\text{d.c.}}_{\delta n \delta n}$ is defined by the general formula (8) for d.c. 'conductivities'. In the second line we have used the fact that to lowest order in energies the quasiparticle susceptibility is only nonzero for fluctuations on the same patch and with the same momentum.

At the end of this section we mention the role of scattering by collective particle-hole excitations that can couple patches. We have also introduced the matrix of relaxation rates Γ, as defined in (8), which is the same as the matrix appearing in (10). These matrices are seen to be the same upon Fourier transforming the conductivity matrix (6) and taking the late time limit [15]. Elasticity of the collisions, together with the fact that Γ becomes independent of the perpendicular momentum to lowest order in energies, allows us to write $\left(\Gamma^{-1}\right)_{\theta\theta' kk'} = \left(\Gamma^{-1}\right)_{\theta\theta'} \delta(\varepsilon_{\theta k} - \varepsilon_{\theta' k'})$. We may therefore use the energy conservation delta function to replace the $\varepsilon_{\theta k}$ in (12) with $\varepsilon_{\theta' k'}$ to obtain

$$\begin{aligned}
\overline{\kappa} &= \frac{1}{T} \int d\theta d\theta' v_{F\,\theta}\, v_{F\,\theta'} \left(\Gamma^{-1}\right)_{\theta\theta'} \int dk' \varepsilon^2_{\theta' k'} \chi_{\delta n_{\theta' k'} \delta n_{\theta' k'}} \int dk\, \delta(\varepsilon_{\theta k} - \varepsilon_{\theta' k'}) \\
&= \frac{T\pi^2}{3} \int d\theta d\theta' v_{F\,\theta}\, v_{F\,\theta'} \left(\Gamma^{-1}\right)_{\theta\theta'} \int dk' \chi_{\delta n_{\theta' k'} \delta n_{\theta' k'}} \int dk\, \delta(\varepsilon_{\theta k} - \varepsilon_{\theta' k'}) \\
&= \frac{T\pi^2}{3} \sigma \,. \qquad (13)
\end{aligned}$$

To obtain the second line we have used the well-known result relating the heat and electric current susceptibilities for free fermions. Namely $\int dk \varepsilon^2 \chi_{\delta n \delta n} = T^2\pi^2/3 \int dk \chi_{\delta n \delta n}$. This follows from performing integrals of the Fermi-Dirac distribution, because the susceptibilities $\chi_{\delta n \delta n} = f'_{\rm FD}(\varepsilon)$, and holds patchwise in a Fermi liquid.

In (13) the thermal conductivity that appears is $\overline{\kappa}$, the thermal conductivity at zero electric field. The thermal conductivity at zero electric current is $\kappa = \overline{\kappa} - \alpha^2 T/\sigma$, with α the thermoelectric conductivity. The thermoelectric conductivity is given by adapting the formulae in (12) in the obvious way. The susceptibility integral that appears is now $\int dk \varepsilon \chi_{\delta n \delta n}$ which, again using the Fermi-Dirac distribution, is seen to go like T^2 at low temperatures. It follows that $\overline{\kappa} \gg \alpha^2 T/\sigma$ at low temperatures, and hence $\kappa = \overline{\kappa}$ in this regime. The result (13) therefore implies the Wiedemann-Franz law (1).

In two space dimensions, Fermi liquid theory suffers from marginally relevant perturbations due to disorder and scattering by collective particle-hole excitations. The collective modes of an interacting disordered Fermi liquid lead, in two dimensions, to logarithmic corrections to the electric and thermal conductivities that grow as the temperature is lowered. These corrections violate the Wiedemann-Franz law, e.g. [27, 28]. In the good metallic regimes we are focussing on, away from metal-insulator transitions, these effects only become important at exponentially small temperatures. Reproducing the effects of scattering by collective particle-hole excitations within our memory matrix approach will presumably require explicit use of the infinitely many conserved densities of interacting Fermi liquid theory as captured via bosonization [29, 30, 31].

The derivation of Wiedemann-Franz that we have given above of course parallels very closely the usual derivation [1]. We have emphasized the role of various almost conserved quantities from the perspective of the effective low energy theory, with a view to generalizing to the non-Fermi liquid case in the following.

4 Non-Fermi liquids without quasiparticles

The essential feature of the strongly interacting non-Fermi liquids we wish to consider is that while the effective low energy theory will still conserve momentum, so that $\dot{\vec{P}} = 0$, the existence of interactions that persist in the low energy theory means that the total electrical and heat currents, $\vec{J}$ and $\vec{Q}$, will neither be conserved nor approximately conserved. We will give examples below. The total electric charge and energy are of course conserved. This is different to Fermi liquids where, due to the fact that interactions have small effects on low energy transport, the currents are also conserved in the low energy theory. Thus, from the outset, the kinematics of conserved quantities underlying non-Fermi liquid transport is significantly different to that of a Fermi liquid. In these strongly interacting and non-quasiparticle based circumstances, we should be wary of overly naïve analogies such as considering scattering by quantum critical bosons to be similar to scattering by phonons.

The conservation of momentum up to effects that are small at low energies is a key assumption that allows the memory matrix method to work. It can be expected to be true in a metal exhibiting a well-defined Drude peak at low temperatures. This is because a perturbative approach to the memory matrix will lead to a sharp Drude peak in (6). Our discussion in this paper does not apply to cases in which momentum-non-conserving interactions have strong effects at low energies. For instance, experimentally speaking, at the boundary of metal-insulator transitions one can encounter 'bad' metals that violate the Mott-Ioffe-Regal resistivity bound [32] and do not exhibit Drude peaks [33].

Strong momentum-violating interactions can manifest themselves in two ways. The first occurs when an emergent particle-hole symmetry in the low energy theory results in susceptibilities such as χ_{JP} going to zero. Then the current can relax quickly without being 'dragged along' by the almost-conserved momentum. The second possibility is that the effects of scattering become so strong that the momentum relaxation rate $\Gamma \to \infty$. These two scenarios are distinguished by the fact that in the first case the spectral weight of the Drude peak vanishes while in the second the weight is conserved but the peak becomes infinitely broad. We briefly discuss these two cases in turn.

Two interesting contexts in which an emergent particle-hole symmetry can arise are in the metallic spin density wave quantum phase transition in two dimensions [34] and in certain theories of continuous Mott transitions [35, 36]. In the spin density wave case, umklapp scattering by finite momentum critical modes connecting the hot spots plays an important role in transport and gives a universal critical conductivity [37]. We shall return to spin density wave transistions in section 5 below, as transport is dominated by excitations away from the hot spots. For the Mott transitions, universal charge transport is obtained from the particle-hole symmetric dynamics of fluctuations about the half-filled state [35, 36]. Particle-hole symmetry is insufficient to obtain universal heat transport, as χ_{QP} will typically not vanish. Therefore heat transport will be tied to the non-universal fate of momentum conservation. We will qualify this statement below, as there are circumstances in which κ is universal even while $\overline{\kappa}$ is not.

A divergent momentum relaxation rate in a theory without disorder is more exotic within conventional theory. What is needed is that an umklapp-like operator become relevant in the low energy theory. This is a priori distinct from Mott physics as it does not depend on charge commensurability. An example of an interaction driven metal-insulator transition with these properties has recently been realized in a holographic context [38].

Having clarified the cases to which our discussion does not apply, we now return to 'good' non-Fermi liquid metals, with a well-defined Drude peak.

While interactions in a non-Fermi liquid mean that the low energy theory is not free, it is still possible that there may be a decoupling of excitations into patches in momentum space, analogous to the patches of a Fermi surface in Fermi liquid theory. In these cases there will not just be one conserved momentum $\vec{P}$, but rather a family of conserved momenta $\vec{P}_\theta$, labelled by the patch θ. We will consider these two cases, with one and with many conserved momenta, separately.

4.1 Hydrodynamic non-Fermi liquid metals

We will refer to metallic systems in which there is only one almost-conserved vector operator in the effective low energy theory, typically the total momentum, as 'hydrodynamic' non-Fermi liquids. For such non-Fermi liquids we can obtain general results concerning the ratio of conductivities. The total momentum is relaxed on a much longer timescale than the currents, schematically $\langle \dot{\vec{P}} \rangle \sim \epsilon \langle \vec{P} \rangle$, with the momentum relaxation rate $\epsilon \ll 1$, while, for instance, $\langle \dot{\vec{J}} \rangle \sim \langle \vec{J} \rangle$. These expressions are written in units of a microscopic energy scale of the system, such as the chemical potential. Clearly the momentum and current operators

are not equal in this example. The only general requirement for our results, however, is that there be only one almost-conserved vector operator. Our discussion is easily adapted to cases in which the electric current is equal to the momentum. From the hierarchy of relaxation rates, we can anticipate that in the $\omega = 0$ memory matrix $M_{PP} \sim \epsilon^2$ and $M_{PJ} \sim M_{PQ} \sim \epsilon$, while the remaining components of the memory matrix (7) are order one. It follows that the inverse of the memory matrix will be dominated by $\left(M^{-1}\right)_{PP} \sim \epsilon^{-2} \gg 1$.

An explicit example of the above logic was discussed in [20]. The effects of a lattice are included in the low energy theory via coupling the theory to an irrelevant operator $\mathcal{O}$ at the lattice momentum k_L: $H = H_0 - \mathcal{O}(k_L)$. It follows that, as operators, $\dot{\vec{P}} = i[H, \vec{P}] = \vec{k}_L \mathcal{O}(k_L)$. In this case one finds that due to the projections $\mathcal{Q}$ in the definition of the memory matrix (7), then $M_{PJ} \sim M_{PQ} \sim \epsilon^2$ are of the same order as M_{PP}, but our general statement above that $\left(M^{-1}\right)_{PP} \sim \epsilon^{-2} \gg 1$ dominates the inverse of the memory matrix remains true [20].

With $\left(M^{-1}\right)_{PP}$ dominating the inverse memory matrix, the d.c. conductivities (8) are[1]

$$\sigma = \chi_{JP}^2 \left(M^{-1}\right)_{PP} , \qquad \overline{\kappa} = \frac{1}{T} \chi_{QP}^2 \left(M^{-1}\right)_{PP} . \tag{14}$$

Taking the ratio of the conductivities in (14), the memory matrix cancels out and we obtain the advertised (2)

$$\frac{\overline{\kappa}}{\sigma T} = \frac{1}{T^2} \frac{\chi_{QP}^2}{\chi_{JP}^2} . \tag{15}$$

The mechanism of momentum relaxation in non-Fermi liquids is an outstanding challenge, manifested for instance in the difficulty in explaining the linear in temperature resistivity of some non-Fermi liquids. It is therefore pleasing that in this formula for the ratio of conductivities the mechanism of momentum relaxation has cancelled out, leaving an expression in terms of purely thermodynamic quantities. In this sense (15) captures a universality analogous to that of the Wiedemann-Franz law. The value and indeed temperature dependence of this ratio will however depend on the low energy theory describing the system.

Transport in theories in which interactions relax currents quickly but not momentum can reasonably be called hydrodynamic, even in low temperature regimes where conventional

[1] To obtain these formulae we have assumed that the different components of the matrix of susceptibilities in (8) do not have strongly differing temperature dependences that could compete with the hierarchy in the components of the inverse memory matrix. The heat current susceptibilities can be expected to have extra powers of (low) temperature relative to the electric current and momentum susceptibilities. These powers of temperature do not work against the hierarchy in the memory matrix leading to (14). If the momentum susceptibility has extra powers of low temperatures relative to e.g. the electric current susceptibilities, such powers compete with our kinematic memory matrix argument above.

hydrodynamics will not apply. It has been suggested in [39, 40] that a similar notion could apply to highly correlated electron gases in semiconductor heterostructures, in temperature regimes where electron interactions equilibrate the system on a parametrically shorter timescale than the timescale for momentum relaxation due to impurities or phonons. Indeed our formula (15) can also be derived, in the relativistic case at least and now at sufficiently high temperatures, using standard hydrodynamics [18]. Hydrodynamic computations will also give an explicit nontrivial temperature dependence of (15) for doped conformal field theories (CFTs) in 2+1 dimensions [18]. We are primarily interested however in the lowest temperature scales, where non-Fermi liquid effects dominate and the memory matrix rather than conventional hydrodynamics provides the appropriate computational framework.

An interesting result is obtained if we consider the thermal conductivity $\kappa = \overline{\kappa} - \alpha^2 T/\sigma$ in this case. Extending the dominance of $\left(M^{-1}\right)_{PP}$ in (14) to the thermoelectric conductivity α, one finds that the two terms in the expression for κ exactly cancel. We explain the simple physics of this cancellation at the end of the following subsection. The leading nonvanishing term in the thermal conductivity is a universal quantity computable in the low energy effective theory – for instance for a doped CFT in 2+1 dimensions, in a temperature regime where hydrodynamics is valid, it is given in terms of the universal electrical conductivity of the CFT [18, 45] – but this universal term does not benefit from the enhancement by $\left(M^{-1}\right)_{PP} \gg 1$. It follows that

$$\frac{\kappa}{\sigma T} \ll 1\,, \tag{16}$$

for these systems. A small Lorenz number therefore seems to be a characteristic feature of hydrodynamic non-Fermi liquid metals. This is perhaps the most dramatic of the kinetically-driven results we will find. At the very least it illustrates how a weakly interacting intuition can be misleading; the result (16) cannot be understood starting from a Fermi liquid and then perturbatively adding the effects of additional neutral degrees of freedom (more heat conduction) or additional inelastic scatterings (more heat relaxation). Rather (16) is a model-independent result for hydrodynamic metals at the lowest temperatures.

Many of the non-Fermi liquid theories that have been explicitly constructed have been based on a patch picture in momentum space, mimicking the low energy structure of a Fermi liquid, and will be considered in the following subsection. Typically these theories, however, are not under complete theoretical control. It seems to be an open question whether in the true IR there is one or many independently conserved momenta. On the other hand, controlled theories with only one conserved momentum at low energy abound in holographic approaches to metals. While some holographic phases of matter do exhibit

intrinsic properties of Fermi surfaces such as logarithmic violation of the boundary scaling of the entanglement entropy [41, 42] or low energy spectral weight at nonzero momentum [43, 44], models considered thus far do not show the large number of conserved densities and momenta associated to a patchlike low energy theory. It is interesting that essential properties of a Fermi surface that occur together in a weakly coupled language – e.g. logarithmic entanglement, finite momentum spectral weight and an infinite number of conserved momenta – can be dissociated in non-quasiparticle based theories.

A difference between our discussion for Fermi liquids and non-Fermi liquids is that in the former case we emphasized the relaxation rates Γ, while for non-Fermi liquids the memory matrix itself is central. This is because for Fermi liquids the single matrix Γ of density relaxation rates controls the relaxation of all quantities. In strongly interacting non-Fermi liquid metals, momentum relaxes via a different mechanism to heat and electric current.

4.2 Non-Fermi liquids with patchwise conserved momenta

If the low energy degrees of freedom of the non-Fermi liquid decouple into patches in momentum space, then there will be many conserved momenta. The difference with the Fermi liquid case will be that the individual patch theories will not be free, and the only almost conserved quantity in each patch will be the momentum $\vec{P}_\theta$. The ratio of conductivities (15) becomes

$$\frac{\overline{\kappa}}{\sigma T} = \frac{1}{T^2} \frac{\int d\theta d\theta' \chi_{Q_\theta P_\theta} \left(M^{-1}\right)_{P_\theta P_{\theta'}} \chi_{P_{\theta'} Q_{\theta'}}}{\int d\theta d\theta' \chi_{J_\theta P_\theta} \left(M^{-1}\right)_{P_\theta P_{\theta'}} \chi_{P_{\theta'} J_{\theta'}}} \,. \tag{17}$$

Part of the assumption of decoupled patches here is that the interpach susceptibilities vanish. This requires that the Landau interpatch interactions are irrelevant in these non-Fermi liquids or that they can be 'diagonalized' to give decoupled patchwise theories. We see immediately that the patchwise susceptibilities entering the above formula are distinct from those appearing in the Fermi liquid case (13). Transport in non-Fermi liquids is controlled by a different underlying kinematical structure. Before attempting to simplify this expression, it is instructive to consider an explicit model of a non-Fermi liquid that falls into the class of theories under consideration.

As an example, we consider the theory of the Ising-nematic quantum phase transition in two dimensional metals developed recently in [46]. Earlier work on this model includes [47, 48, 49]. While the loop expansion is ultimately not controlled in this model, it will serve to illustrate the different ways in which various quantities relax. The theory is obtained by zooming in on a pair of antipodal patches of a Fermi surface and maintaining a scaling regime

where interactions with a collective boson are strong. Because the fermions only interact efficiently with the boson when the boson momentum is parallel to the Fermi surface, due to the Fermi surface curvature, to capture this process it is necessary to 'thicken' the patches. Thus, unlike in the Fermi liquid case, the patchwise theory describes propagation in two space dimensions [46]. The effective patchwise low energy theory of [46] describes two antipodal fermions ψ_s, with $s = \pm$, interacting with a gapless boson ϕ

$$\mathcal{L} = \sum_{s=\pm} \psi_s^\dagger \left(i\eta\partial_t + is\partial_x + \partial_y^2 \right) \psi_s - \sum_{s=\pm} \lambda_s \phi \psi_s^\dagger \psi_s + \frac{N}{2e^2} \left[\eta' \left(\partial_t \phi \right)^2 - \left(\partial_y \phi \right)^2 \right] . \tag{18}$$

The spin flavor index running from 1 to N has been suppressed. In the Ising-nematic model $\lambda_+ = \lambda_-$. This theory can also describe a spin liquid when $\lambda_+ = -\lambda_-$. The renormalization group (RG) scaling that one considers for this model is: $\partial_x \to b^2 \partial_x, \partial_y \to b\partial_y, \partial_t \to b^3 \partial_t$ together with $\psi \to b^2 \psi, \phi \to b^2 \phi$. The couplings $\lambda_\pm$ and e are dimensionless under this scaling. We see that both of the time derivative terms we have included in the above Lagrangian are in fact irrelevant under this scaling. Thus we should consider $\eta, \eta' \to 0$. Nontrivial frequency dependence compatible with the RG scaling will be generated radiatively [46]. We have included the time derivative terms above in order to be able to easily define tree level operators for e.g. the momentum. The fact that the classical frequency dependence will be swamped at low energies by terms generated in the RG flow tells us that the momentum operators we will shortly define in fact undergo very significant vertex corrections, which we will ignore. This is one of several reasons why the comments that follow should be taken at the level of a qualitative discussion of momentum versus current relaxation in these theories.

We proceed to compute the time derivatives of currents and momenta in the patch theory (18). These expressions could be used to perform computations to estimate the temperature dependence of the quantities appearing in the conductivity (8). Our more modest goal here is to use these expressions as an explicit example of how patchwise momenta but not currents are conserved in a strongly interacting theory. Given the Lagrangian (18) we can write down the patch momenta and current densities using the Noether procedure and the equations of motion

$$\vec{p} = \sum_s \frac{i\eta}{2} \left(\vec{\nabla} \psi_s^\dagger \, \psi_s - \psi_s^\dagger \vec{\nabla} \psi_s \right) + \frac{N}{e^2} \eta' \, \partial_\tau \phi \vec{\nabla} \phi \, , \tag{19}$$

$$j_x = \sum_s s \psi_s^\dagger \psi_s \, , \tag{20}$$

$$j_y = i \sum_s \left(\partial_y \psi_s^\dagger \, \psi_s - \psi_s^\dagger \partial_y \psi_s \right) \, , \tag{21}$$

$$q_x = \frac{-1}{2\eta}\sum_s s\left(\psi_s^\dagger(is\partial_x+\partial_y^2)\psi_s + \text{c.c}\right) + \frac{1}{\eta}\sum_s \lambda_s\,\phi\,\psi_s^\dagger\psi_s\,, \tag{22}$$

$$q_y = -\frac{1}{\eta}\sum_s\left(\partial_y\psi_s^\dagger(-\partial_x+i\partial_y^2)\psi + \text{c.c}\right) - \frac{N}{e^2}\partial_y\phi\,\dot{\phi} + \frac{1}{\eta}\sum_s \lambda_s\phi\left(i\partial_y\psi_s^\dagger\psi_s + \text{c.c}\right)\,, \tag{23}$$

and the Hamiltonian density is

$$h = \frac{N}{2e^2}\left[\eta'\left(\partial_\tau\phi\right)^2 + \left(\partial_y\phi\right)^2\right] + \sum_s\left(\frac{is}{2}\left[\partial_x\psi_s^\dagger\psi_s - \psi_s^\dagger\partial_x\psi_s\right] + \partial_y\psi_s^\dagger\partial_y\psi_s + \lambda_s\phi\psi_s^\dagger\psi_s\right)\,. \tag{24}$$

The total patch quantities $\{\vec{P}, \vec{J}, \vec{Q}, H\}$ are obtained by integrating the densities over space. This allows us to compute the time derivatives

$$i\dot{\vec{P}} = [P, \vec{H}] = 0\,, \tag{25}$$

$$i\dot{J}_x = [J_x, H] = 0\,, \tag{26}$$

$$i\dot{J}_y = [J_y, H] = \frac{-2i}{\eta}\sum_s \lambda_s\int d^2x\,\partial_y\phi\,\psi_s^\dagger\psi_s\,, \tag{27}$$

$$i\dot{Q}_x = [Q_x, H] = \frac{i}{\eta}\sum_s \lambda_s\int d^2x\,\dot{\phi}\psi_s^\dagger\psi_s\,, \tag{28}$$

and

$$i\dot{Q}_y = \sum_s\int d^2x\,\left(\left[-\frac{i\lambda_s}{\eta'}\phi + \frac{i\lambda_s^2}{\eta^2}\phi^2\right]\partial_y(\psi^\dagger\psi) + \frac{i\lambda_s\phi}{\eta^2}\left[\left(\partial_y^3\psi^\dagger\psi - \partial_y\psi^\dagger\partial_y^2\psi\right) + \text{c.c.}\right] + \frac{2ie^2\lambda_s}{N\eta\eta'}\dot{\phi}\left[i\partial_y\psi^\dagger\psi + \text{c.c.}\right] + \frac{is\lambda_s}{\eta^2}\phi\,\partial_x\left[i\partial_y\psi^\dagger\psi + \text{c.c.}\right]\right)\,. \tag{29}$$

This exercise shows explicitly how interactions cause currents to relax while conserving momentum. The coupling constants are order one at the fixed point. The fact that the electrical current perpendicular to a given patch on the Fermi surface, J_x, is conserved is probably not a complication – for an electric field applied in some given direction, this fact applies to a set of excitations of measure zero on the Fermi surface. As we already noted above, strong vertex corrections will mean that obtaining the actual relaxation of currents from the above formulae will require a nontrivial (but doable) diagrammatic computation.

The expression (17) for the ratio of conductivities is less universal than our result (15) for hydrodynamic metals. Let us assume that the irrelevant interpatch scattering is controlled by one scale to leading order at low energies, so that we can write $\left(M^{-1}\right)_{P_\theta P_{\theta'}} = \lambda\,F(\theta,\theta')$. Here λ is a rate of momentum relaxation whereas $F(\theta,\theta')$ is a dimensionless 'kinematic' function of pairs of points on the Fermi surface that does not contain another scale. Then

we can write, at least in terms of extracting the temperature dependence of the ratio,

$$\frac{\overline{\kappa}}{\sigma T} \sim \frac{1}{T^2} \left\langle \frac{\chi_{PQ}^2}{\chi_{PJ}^2} \right\rangle \sim \frac{\kappa}{\sigma T} \,. \tag{30}$$

The angled brackets here denote a schematic average over the Fermi surface in which the relaxation rate λ has cancelled out.

To obtain the second relation in (30), we can first explain why the cancellation we found above for hydrodynamic metals in the computation of $\kappa = \overline{\kappa} - \alpha^2 T/\sigma$ does not occur here. Recall that κ is defined as the heat conductivity at vanishing electric current. Because $\chi_{JP} \neq 0$ for the metallic states we are considering, the no-current boundary condition requires that the total momentum also vanish. For the hydrodynamic metals, the total momentum was the only conserved quantity. Therefore, in states with vanishing total momentum, the heat current can relax and heat conduction is universal. In the metals with patchwise conserved momenta, however, the vanishing of the total current does not imply that all of the patch momenta must independently vanish. The nonvanishing patch momenta will then not allow the total heat current to relax within the momentum-conserving low energy effective theory. This can be seen explicitly by verifying that because of the integrals over patches on the Fermi surface in expressions like (17), the $\left(M^{-1}\right)_{P_\theta P_{\theta'}}$ dependence no longer cancels out in $\kappa = \overline{\kappa} - \alpha^2 T/\sigma$. In addition to the absence of a cancellation, we can argue that $\kappa \sim \overline{\kappa}$ have the same temperature scaling: In these non-Fermi liquids where only the patchwise momenta are conserved, only χ_{PJ} and χ_{PQ} appear in the conductivities. It is then easily seen that, unlike in the Fermi liquid case, $\alpha^2 T/\sigma$ and $\overline{\kappa}$ contain exactly the same susceptibilities and therefore have the same temperature dependence. Thus we finally obtain the second relation in (30).

Universal heat conduction has also been obtained theoretically in the past by dividing the system into two sets of modes. Among the first set of modes, interactions degrade all currents while the momentum can be transferred to the second set of modes. The second set of modes is then assumed to dissipate momentum very quickly. This kind of hierarchy can be formalized to some extent via large N expansions. In these circumstances a universal heat conduction can be associated to the first set of modes. See for instance [50, 51]. Such hierarchies are perhaps more natural in the context we consider in the following section, in which only a subset of the degrees of freedom goes critical (at e.g. hot spots or in a single quantum critical band). The remaining degrees of freedom are long-lived, but can dump momentum and energy into the critical modes.

5 Non-Fermi liquids with (some) long-lived quasiparticles

In many experimental systems believed to be close to quantum critical points, the order parameter carries a finite wavevector. This occurs for instance in metallic spin and charge density wave transitions. In such cases, fluctuations of the order parameter are most efficient at scattering low energy fermions in the vicinity of hot spots or hot lines on the Fermi surface. The hot loci are connected in momentum space by an ordering wavevector while the remaining patches on the Fermi surface are referred to as 'cold'. The cold fermions can in turn scatter off excitations at the hot loci [37]. There are three logical possibilities: (i) All quasiparticles, both hot and cold, are destroyed by quantum critical and interpatch scattering, (ii) the hot degrees of freedom are strongly interacting while the cold quasiparticles remain long lived and (iii) the hot and cold fermions both remain long lived, with possibly distinct lifetimes. The first possibility falls under the remit of the previous section 4. We proceed to discuss the remaining cases, in which at least some long-lived quasiparticles survive in the regime showing non-Fermi liquid transport.

The possibility of long-lived quasiparticles coupled to strongly interacting critical excitations also arises in systems with multiple bands. One band can become 'hot' while the other 'cold' bands retain a quasiparticle character. Evidence for this phenomenon can be seen in quantum oscillation experiments in e.g. $Sr_3Ru_2O_7$ [52]. Interband scattering has the potential to lead to unconventional lifetimes for the stable quasiparticles.

In terms of our effective field theory approach to quantum transport, in the cases where quasiparticle and non-quasiparticle excitations coexist, one must consider separately the strongly interacting patches at or close to the hot loci and the quasiparticle patches away from the hot loci. For the long-lived cold fermions, we can apply the memory matrix method much as for Fermi liquids. As we noted above, in spin and charge density wave transitions in two space dimensions, the theory of the hot loci involves strong scattering with a nonzero momentum transfer [37]. Momentum is therefore not approximately conserved and the memory matrix method may not be applicable to the hot excitations. On the other hand, this fact also suggests that once momentum is transferred from the cold patches to the hot loci via weak interpatch scattering, then it will be relaxed very quickly by the hot excitations. We will discuss the consequences of this observation shortly.

The interplay of hot and cold patches has immediate consequences for transport. An important likelihood is that any anomalously large resistivity from the hot patches will be short circuited by the cold fermions [53]. The manner in which this effect occurs depends on the almost-conserved quantities in the system. We consider two scenarios in turn.

In the first scenario, both the hot and cold excitations retain a quasiparticle character. There are then two scattering rates $\tau_{\rm hot}^{-1}$ and $\tau_{\rm cold}^{-1}$. While both flavors of quasiparticle experience inelastic scattering by critical excitations and by other electronic quasiparticles, they also both experience elastic impurity scattering. For this scenario we assume that the temperature dependence of the cold quasiparticle lifetime is the conventional $\tau_{\rm cold}^{-1} \sim x + T^2$. A model along these lines was considered in [54] to describe transport near the spin density wave transition in three dimensional heavy fermion compounds. At the higher of the low temperatures of interest, inelastic scattering dominates and the hot modes are indeed short-circuited by the cold fermions, leading to conventional transport. At intermediate low temperatures however, with $T^2 < x$, the effect of impurity scattering on the cold quasiparticles can prevent the short-circuiting. In this regime an unconventional temperature dependence of the conductivity obtains, as the hot quasiparticles contribute to transport and $\tau_{\rm hot}^{-1}$ is not necessarily dominated by impurity scattering. For instance if $\tau_{\rm hot}^{-1} \sim x + T^\alpha$ with $\alpha < 2$. At the very lowest temperatures, both hot and cold modes are scattered predominantly by impurities. These elastic scatterings will obey the Wiedemann-Franz law as per our discussion of Fermi liquids in section 3 above. Furthermore, any additional contribution to the heat conductivity, for instance from spin fluctuations, can be expected to be small at these very low temperatures because of their small specific heat at low temperature and possibly also because they are able to relax their momentum efficiently through e.g. lattice scattering.

The model of the previous paragraph shows one way in which the hot versus cold dichotomy allows the Wiedemann-Franz law to hold at the lowest temperatures consistently with non-Fermi liquid transport at slightly higher temperatures. Crossovers from non-Fermi liquid transport to Wiedemann-Franz at low temperatures have been observed in several materials that we discuss further in section 6 below. While we focused on hot versus cold modes above, these crossovers are also obtained if all quasiparticles are long lived but with unconventional lifetimes $\tau^{-1} \sim x + T^\alpha$.

We now turn, in more depth, to situations in which quasiparticles are destroyed at the hot loci and furthermore the cold fermions acquire non-Fermi liquid lifetimes through scattering off hot excitations, while remaining well-defined quasiparticles. In distinction to the model described in the previous two paragraphs, we will see that such scenarios allow the Wiedemann-Franz law to coexist at the same low temperatures as non-Fermi liquid transport is observed.

5.1 Linear in temperature resistivity and Wiedemann-Franz

The previous subsection described situations in which intermediate low-temperature non-Fermi liquid behavior could cross over to Fermi liquid and Wiedemann-Franz obeying transport at the lowest temperatures. In this section we discuss the prospects for satisfying the Wiedemann-Franz law simultaneously with 'non-Fermi liquid' transport. Such coexistence is also observed in several materials, as we discuss in section 6 below. Indeed, if the Wiedemann-Franz law is observed to be satisfied over a range of temperatures exhibiting unconventional transport, this places strong constraints on the dissipative process controlling the transport.

It is well known that for temperatures above the Debye temperature but below the Fermi energy, scattering of electrons by phonons leads to a linear in temperature resistivity and also to the Wiedemann-Franz law [1]. There are four important facts in this temperature range that allow these two phenomena to coexist: (i) the rate at which the electronic quasiparticles lose momentum to the phonons is given by $\Gamma \sim T$, (ii) the phonons lose momentum via Umklapp scattering at a faster rate $\Gamma_U \gg \Gamma$, (iii) the electronic quasiparticles remain long lived and (iv) because the average energy of the fermions involved in the scattering is much larger than that of the phonons (whose energy is bounded above by the Debye energy), the electrons effectively experience elastic scattering. It is clear that points (iii) and (iv) allow the derivation of Wiedemann-Franz in section 3 to go through. Points (i) and (ii) are additionally required because they ensure that electron scattering immediately leads to momentum dissipation, allowing currents to relax. Point (ii) is furthermore related to the fact that the electronic contribution to thermal conduction will be much bigger than the phonon contribution.

It is interesting to ask what general types of emergent collective low energy excitations could satisfy conditions (i) to (iv). To connect with some of the more challenging experimental data, with linear resistivity observed at very low temperatures, the effective Debye temperature should be very low or even vanishing. This question has acquired a certain urgency following the observation that the measured linear in temperature scattering rate shows a degree of universality across a range of different materials [55]. In some of these unconventional materials, a strong case can be made that the linear in temperature resistivity is due to scattering of well-defined quasiparticles by critical modes that are classicalized by being above their effective Debye temperature [55].

It is instructive to take a somewhat general approach. Suppose we are given a critical bosonic mode $\mathcal{O}$ with a retarded Green's function $D^R(\omega, k)$. Let the scattering of the cold

fermions ψ with the bosonic mode be described by the coupling $S_{\rm int} = \lambda \int dt d^d x \psi^\dagger \psi \mathcal{O}$. To describe scattering by hot fermions, we can imagine that the operator $\mathcal{O}$ is a fermion bilinear. We assume that this coupling can be treated perturbatively at low energies, consistent with the survival of the cold fermions as quasiparticles. This kind of coupling between cold fermions and hot operators was considered in [37] for the metallic spin density wave transition at zero temperature. If we assume that the hot excitations are able to dissipate momentum very efficiently, as seems to be the case for the spin density wave transition in two dimensions [37], then the transport will be dominated by cold quasiparticles. This transport will be controlled by the decay rate

$$\mathrm{Im}\Sigma(\omega, p) = \lambda^2 \int \frac{d^d k}{(2\pi)^d} \frac{d\Omega}{\pi} \mathrm{Im}G^R(\omega - \Omega, p - k)\, \mathrm{Im}D^R(\Omega, k) \frac{f_{\rm FD}(\omega - \Omega) f_{\rm BE}(\Omega)}{f_{\rm FD}(\omega)} \,. \tag{31}$$

This formula is essentially Fermi's golden rule allowing for decay into non-quasiparticle modes and is easily derived using standard thermal field theory methods. Here $f_{\rm FD}$ and $f_{\rm BE}$ are the Fermi-Dirac and Bose-Einstein distributions, respectively. In general we should allow for the coupling λ to be k dependent. For instance, actual phonons can couple derivatively to the fermions, which is important for scattering below the Debye scale. The free fermion Green's function is $G^R(\omega, k)$. We may simplify this expression using the fact that for the free fermion

$$\mathrm{Im}G^R(\omega, k) = \pi\, \delta(\omega - \epsilon(k)) \,, \tag{32}$$

where $\epsilon(k)$ is the free fermion dispersion and vanishes on the Fermi surface. We can now ask: for what Green's functions $D^R(\omega, k)$ is a linear in temperature resistivity obtained? When is this scattering sufficiently elastic that the Wiedemann-Franz law is true?

Elasticity will hold when the energy transfer is much less than the typical energy of the cold electrons, namely when $\Omega \ll \omega \sim T$ throughout the integrand in (31). The fermion energy ω is measured from the chemical potential. From (31), expanding the factor of $f_{\rm BE} \approx T/\Omega$ and noting the cancellation of the two factors of $f_{\rm FD}$ in this regime of energies, we obtain

$$\mathrm{Im}\Sigma(\omega, p) = \lambda^2 T \int \frac{d^d k\, d\Omega}{(2\pi)^d} \frac{\mathrm{Im}D^R(\Omega, k)}{\Omega} \delta(\Omega - \omega + \epsilon(p - k)) \,. \tag{33}$$

If the bosonic spectral weight $\mathrm{Im}D^R$ does not have a strong temperature dependence, and dies off sufficiently quickly at large energies so that there is no contribution to the integral from $\Omega \sim T$, then we immediately obtain a linear in temperature scattering rate. For instance, for Debye phonons,

$$\mathrm{Im}D^R_{\rm phon.}(\Omega, k) = \delta(\Omega^2 - c_s^2 k^2) \,. \tag{34}$$

The lattice cutoff on the momentum $k < k_L$ therefore implies that $|\Omega| < c_s k_L$. It follows that if the temperature is above the Debye temperature, $c_s k_L \ll T$, we consistently find elastic scattering with a linear in temperature decay rate. This is of course a textbook result for high temperature scattering of electrons by phonons.

Given that the boson in the cases of interest is emerging from a strongly correlated sector, we do not expect its spectral weight to take the free form (34). We will call a bosonic mode with a spectral weight $\mathrm{Im}D^R(\Omega, k)$ such that there exists an effective Debye scale Ω_D, so that for $T \gg \Omega_D$ equation (33) holds, a 'generalized phonon'. Generalizing the spectral weight of the mode will be important for our discussion of thermodynamics around equation (36) below.

If classicalized phonons provide one canonical way to obtain a linear in temperature scattering rate via a bosonic mode, the other prototypical framework is the marginal Fermi liquid [56]. There the bosonic spectral weight is postulated to have the form

$$\mathrm{Im}D^R_{\mathrm{MFL}}(\Omega, k) \sim \begin{cases} \Omega/T & \Omega \lesssim T \\ \mathrm{sgn}(\Omega) & \Omega \gtrsim T \end{cases}, \tag{35}$$

over some large range of momentum. Setting $\Omega = T\hat{\Omega}$ and $\omega = T\hat{\omega}$ in the formula for the lifetime (31) and using (32) we again obtain a linear in temperature relaxation rate. This is a completely different regime from the phonon scattering. The energy transferred to the bosonic mode saturates the temperature scale and hence the scattering is manifestly not elastic. Independently of how efficiently the boson can lose its momentum and the extent to which the boson contributes to thermal transport, the mechanism is not compatible with the Wiedemann-Franz law. This is because inelasticity means that we have lost the simple relation between current and heat relaxation that played a key role the derivation of the law in section 3.

The Wiedemann-Franz law therefore is a diagnostic that can differentiate marginal Fermi liquid-like from phonon-like linear in temperature relaxation rates.

As is well known, the fermion relaxation rate does not directly give the transport relaxation rate. In particular these quantities can differ if small-angle scattering dominates the transport. However, because we are considering temperatures above the effective Debye temperature of the 'generalized phonon' mode, typical scatterings will involve a momentum transfer of order the lattice momentum. These large momentum scatterings relax currents and momenta at the same rate as the fermions relax. It is therefore sufficient to consider the quasiparticle decay rate (31) to obtain the d.c. resistivity.

Several materials showing a linear in temperature resistivity at low temperatures also show a specific heat with a temperature dependence of $c \sim -T \log T$, e.g. [57, 58]. This poses a problem if the quantum critical mode behind the linear in temperature resistivity is literally a free phonon, with spectral weight (34), as it is well known that such phonons will contribute a constant specific heat over the regime of interest, contradicting the observations.[2] However, we do not expect the critical mode to be a free boson. The contribution of a 'generalized phonon' with retarded Green's function $D^R(\omega, k)$ to the specific heat may be computed from the following general expression for the entropy density

$$s = \int \frac{d^d k}{(2\pi)^d} \frac{d\Omega}{\pi} \frac{\Omega}{8T^2} \frac{\text{Im} \log D^R(\Omega, k)}{\sinh^2 \frac{\Omega}{2T}}, \tag{36}$$

$$= \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{d\Omega}{\pi} \frac{\Omega}{4T^2} \frac{\arg D^R(\Omega, k)}{\sinh^2 \frac{\Omega}{2T}}. \tag{37}$$

This formula is obtained from standard thermal field theory manipulations, using a spectral representation for $\log D^R(\omega, k)$. The specific heat is then given as usual by $c = T\partial s/\partial T$. To obtain the second line we used the fact that the imaginary part of the retarded Green's function is odd and the real part even, so that $D^R(-\Omega, k) = \overline{D^R(\Omega, k)}$.

Now consider a generalized phonon with spectral weight satisfying

$$\text{Im}\, D^R_{\text{g-phon.}}(\Omega, k) = 0 \qquad \text{for} \qquad |\Omega| > \Omega_\star(k)\,. \tag{38}$$

This is the simplest way to implement a Debye scale $\Omega_D = \max_k \Omega_\star(k)$. We are assuming nothing about the distribution of spectral weight at energies below $\Omega_\star(k)$. From the Krammers-Kronig relation we obtain the following result for the real part

$$\text{Re}\, D^R_{\text{g-phon.}}(\Omega, k) < 0 \qquad \text{for} \qquad \Omega > \Omega_\star(k)\,. \tag{39}$$

This result uses only positivity of the spectral weight (for $\Omega > 0$). It follows that for $\Omega > \Omega_\star(k)$ the argument in (37) satisfies $\arg D^R(\Omega, k) = \pi$. In the regime of temperatures where we obtain a linear in temperature scattering rate, $T \gg \Omega_D$, we can now isolate the following universal contribution to the entropy density

$$s_{\text{univ.}} = \int \frac{d^d k}{(2\pi)^d} \int_{\Omega_\star(k)}^T \frac{d\Omega}{\Omega} = \int \frac{d^d k}{(2\pi)^d} \log \frac{T}{\Omega_\star(k)}\,. \tag{40}$$

This gives the constant specific heat

$$c = \int \frac{d^d k}{(2\pi)^d}\,, \tag{41}$$

[2] This point was emphasized to us by Andy Mackenzie.

as we might have anticipated from energy equipartition. Thus we find that the simplest generalized phonons of (38) require an (unobserved) constant contribution to the specific heat at low temperatures. One way out may be that the generalized phonons are only active over a restricted range of momenta, allowing the contribution (41) to the specific heat to be sufficiently small as to be unobserved. This may be natural if the generalized phonon is associated with hot spots or narrow bands. This observation motivates future consideration of a detailed microscopic model of generalized phonons, as well as experimental probes that might be able to resolve a small constant contribution to the specific heat at low temperatures in materials showing a linear in T resistivity with Wiedemann-Franz simultaneously satisfied.

6 Revisiting the experiments

Following the discussions above of various flavors of non-Fermi liquids with and without quasiparticles, it is instructive to revisit the experimental results.

First, consider together the heavy fermions $YbRh_2Si_2$, $CeCoIn_5$, $CeRhIn_5$ and the ruthenate $Sr_3Ru_2O_7$. These materials exhibit pronounced non-Fermi liquid behavior at, or in the vicinity of, a metallic critical point. $YbRh_2Si_2$ and $Sr_3Ru_2O_7$ exhibit field-tuned magnetic quantum critical points (QCPs), the former being between an antiferromagnet and a paramagnetic Fermi liquid state, while the latter exhibits a metamagnetic transition. $CeRhIn_5$ is an incommensurate antiferromagnet below a critical temperature $T_N = 3.8K$; above T_N, its transport displays pronounced effects due to spin fluctuations, in a way reminiscient of the other metallic QCPs. Its close cousin $CeCoIn_5$ exhibits a field-tuned superconductor to Fermi liquid QCP, together with strongly fluctuating antiferromagnetism. In the following discussion, we will further summarize the relevant transport phenomenology for these systems.

In applying our considerations to these materials we are implicitly assuming that they show well-defined Drude peaks. To our knowledge, for none of these materials has the optical conductivity been measured at the critical magnetic fields. Such measurements would certainly be of interest. In the absence of magnetic fields or pressure, the heavy fermion optical conductivities have been studied in e.g. [59, 60] where a sharp Drude peak was observed over temperature ranges showing non-Fermi liquid transport.

In the discussion below we ignore the direct effects of magnetic fields on transport. Magnetic fields can be incorporated into the memory matrix formalism and have a strong effect

on the d.c. conductivities if they dominate over other sources of momentum relaxation, e.g. [18]. We have estimated that the energy scale associated to the critical field is comparable to the momentum relaxation rate in the experiments. The direct effects of magnetic fields on strongly correlated transport deserve further study both theoretically and experimentally. The ratio of electric and thermal Hall conductivities in $CeCoIn_5$ was studied in [61].

A plot of the ratio of the electronic contribution to the conductivities $L/L_0 \equiv \kappa/\sigma T \times 3/\pi^2$ as a function of low temperature in $CeCoIn_5$, $CeRhIn_5$ and $Sr_3Ru_2O_7$ [4, 13, 6, 62] shows the same shape as is observed in conventional metals due to scattering off phonons and impurities [1]. That is, there are three regimes: At the highest temperatures, Wiedemann-Franz is obeyed due to scattering by phonons above their Debye temperature, as we recalled in the previous section. Below the Debye temperature, Wiedemann-Franz is violated due to the onset of inelastic scattering by phonons. At the lowest temperatures, Wiedemann-Franz is again recovered due to elastic impurity scattering dominating. We sketch this situation in figure 1 below. For the non-Fermi liquids, however, the temperature scales at which these transitions occur are well below the Debye scale of the metals and therefore are presumably not related to actual phonon scattering.

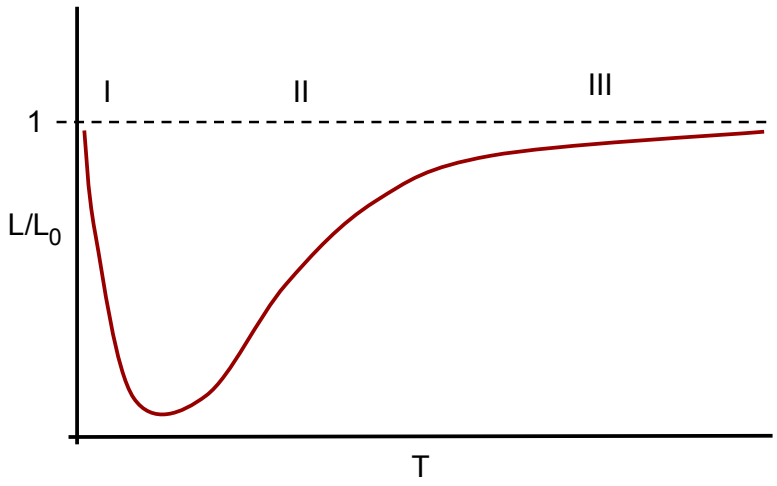

Figure 1: Ratio of conductivities as a function of temperature in conventional metals and in certain non-Fermi liquids. Wiedemann-Franz is satisfied in regions I and III, but not in the intermediate region II. In conventional metals the crossover from II to III occurs at the Debye temperature. The non-Fermi liquids under discussion exhibit the same structure at very low temperatures. The non-Fermi liquids may in addition show a conventional crossover at their Debye temperature, that is not related to our discussion here.

It is natural to ascribe the Wiedemann-Franz law at the higher of these low temperature regimes in the non-Fermi liquids to well-defined quasiparticles scattering elastically off a quantum critical 'generalized phonon' mode of the sort we described in section 5 above. This notion is further supported by the temperature dependence of the electrical resistivity in these three different regimes. The relevant experimental information is summarized in table 1 below.

Material	region I: WF very low T	region II: ~~WF~~ intermediate low T	region III: WF higher low T
$CeCoIn_5$ (a-axis, critical field) [4, 13]	yes	crossover to $\rho \sim T^{3/2}$	$\rho \sim T$
$CeRhIn_5$ (non-critical field, AFM) [62]	yes	$\rho \sim T^2$	$\rho \sim T$
$Sr_3Ru_2O_7$ (non-critical field) [6, 55, 58]	yes	$\rho \sim T^2$	$\rho \sim T$
$Sr_3Ru_2O_7$ (critical field) [6, 55, 58]	yes	$\rho \sim T$	$\rho \sim T$
$YbRh_2Si_2$ (critical field) [63, 7, 8, 9]	disputed	$\rho \sim T$	no measurements
$CeCoIn_5$ (c-axis, critical field) [13]	no	$\rho \sim T$	no measurements

Table 1: Schematic electrical resistivity in the three low temperature regimes of figure 1, characterized by whether or not Wiedemann-Franz holds, for several non-Fermi liquid materials. In the description of the materials, 'critical field' means that the material has been tuned to a quantum critical point by an external magnetic field. 'No measurements' means that a higher temperature region III with Wiedemann-Franz satisfied is not found at the temperatures currently reported. The materials can be divided into those in which the linear resistivity is observed simultaneously with Wiedemann-Franz holding and those in which it is not. This suggests a division into non-Fermi liquids that do not and do, respectively, admit a quasiparticle description of transport, as we discuss in the text. Critical $Sr_3Ru_2O_7$ is a confusing exception here, as we also discuss in the text.

Non-critical $Sr_3Ru_2O_7$, $CeRhIn_5$ and critical a-axis $CeCoIn_5$

Without considering the microscopics of the different materials in table 1, in particular the various ordered phases that appear in their phase diagrams, it is possible to make some simple observations. Three cases show a crossover from Wiedemann-Franz together with linear in temperature resistivity above some temperature to violation of Wiedemann-Franz together with a resistivity scaling like $\rho \sim T^x$, with $x > 1$, below that temperature ($Sr_3Ru_2O_7$ and $CeRhIn_5$ away from criticality and a-axis transport in $CeCoIn_5$ at criti-

cality). This is exactly what one expects for scattering of well-defined quasiparticles by a 'generalized phonon' mode going through its effective 'Debye temperature'. The onset of inelasticity below the Debye temperature means that the heat conductivity becomes smaller and hence L/L_0 drops below one, while the resistivity is no longer dominated by scattering off classicalized modes and so is no longer linear in temperature. Restoration of Wiedemann-Franz at the lowest temperatures in these materials is then also to be expected because the electronic quasiparticle has not been destroyed and therefore we can expect elastic scattering from impurities to dominate at the lowest temperatures, as described in section 3. Finally, the crossover in the resistivity and Wiedemann-Franz in a-axis transport in $CeCoIn_5$ at criticality indicates that not all scales, in particular the effective Debye scale, have collapsed at the critical point [4].

The behavior of these three materials is consistent with the kinematical framework of quantum criticality combined with long-lived particles described in section 5 above.

Critical $YbRh_2Si_2$, c-axis $CeCoIn_5$ and $Sr_3Ru_2O_7$

In critical $YbRh_2Si_2$ and $Sr_3Ru_2O_7$ and c-axis transport in critical $CeCoIn_5$, in contrast, a linear in temperature resistivity is found in temperature regimes in which Wiedemann-Franz is not satisfied. The fact that $L/L_0 < 1$ in this regime implies that, if a quasiparticle description is valid, then the violation is due to inelastic scattering (relaxing heat current faster than charge current) rather than due to additional heat carriers. Given that the linear in temperature resistivity in this regime would therefore arise from inelastic scattering, it cannot be due to scattering by a generalized phonon at temperatures above its Debye frequency, which is effectively elastic. This transport cannot therefore be explained by a Debye scale collapsing to zero at the critical point. The alternatives are that either the linear in temperature resistivity is due to a (necessarily inelastic) marginal-Fermi liquid type of scattering, or there are no quasiparticles and the transport is strongly correlated and described by the kinematics of section 4 above. The latter is an interesting possibility for $YbRh_2Si_2$ and c-axis $CeCoIn_5$, over this range of temperatures.

It is hard to understand, however, how in $Sr_3Ru_2O_7$ tuned to criticality Wiedemann-Franz can be restored at higher temperatures without even the slope in the linear resistivity changing! This is because the linear resistivity would seem to be necessarily due to different scattering mechanisms in the two regimes (i.e. with and without Wiedemann-Franz holding). We do not know the resolution to this puzzle. One possibility is that the observations of thermal conductivity in $Sr_3Ru_2O_7$ in [6] predated the more precise measurements of elec-

trical conductivity in [58, 55]. Perhaps the thermal conductivity in the critical regime should be revisited. It is tempting to predict that revisited measurements will see the Wiedemann-Franz law hold down to low temperatures at the critical magnetic field (at least until the nematic phase intervenes). There is some evidence in [6] that the effective Debye scale may be collapsing to zero as the critical field is approached, allowing Wiedemann-Franz and linear resistivity to jointly hold down to lower temperatures.

We have noted that existing data for $YbRh_2Si_2$ and c-axis $CeCoIn_5$ may be compatible with strongly interacting transport without quasiparticles, at least over the range of temperatures showing linear resistivity with Wiedemann-Franz violated. If this is the case we can expect formulae like (30) to hold and describe the violation of the Wiedemann-Franz law. Interestingly, in both these materials, over the low temperature range showing violation of Wiedemann-Franz, the Lorenz ratio has a simple temperature dependence. In c-axis $CeCoIn_5$ it is almost independent of temperature [13] while in $YbRh_2Si_2$ it has a weakly linear dependence on temperature [7, 8]. This is suggestive of the notion that these dependences could be obtained from a ratio of susceptibilities such as (30). In particular, a constant L/L_0 as a function of temperature is obtained under the seemingly reasonable assumption that $\chi_{QP} \sim T\,\chi_{JP}$, while a linear temperature dependence would require $\chi_{QP} \sim T^{3/2}\,\chi_{JP}$. If this is true, we can predict that L/L_0 will continue to be constant or linear at larger temperatures, while the linear in temperature resistivity holds, and Wiedemann-Franz will not be recovered.[3] It would be interesting to measure $\overline{\kappa}$ in these materials, where any universality may be expected to be more pronounced. In fact, the temperature scaling $\kappa \sim \overline{\kappa}$ without strict equality (recall that $\kappa = \overline{\kappa}$ for quasiparticles) is a diagnostic for strongly correlated transport with patchwise conserved momenta. If transport in these materials is indeed strongly correlated, a microscopic approach to the transport must use the kinematic framework of almost-conserved charges developed in earlier sections of this paper, rather than a language of elastic versus inelastic scattering or of additional neutral charge carriers.

The possible restoration of Wiedemann-Franz at the lowest temperatures in $YbRh_2Si_2$

[3] As this paper was completed, measurements of the Lorenz ratio in YbAgGe were announced [64]. At the critical magnetic field, these showed L/L_0 depending linearly on temperature over a range of low temperatures, violating Wiedemann-Franz and furthermore crossing straight through the Wiedemann-Franz value. If the thermal transport is not dominated by phonons – see the caveats described in [64] – then these results are potentially a striking realization of strongly correlated transport as we have characterized it in section 4: in the critical regime, showing a linear in temperature electrical resistivity, the Wiedemann-Franz value for the ratio of conductivities does not appear as a useful reference point for understanding the transport.

at the critical magnetic field [8, 9] would indicate that there is an energy scale that is not collapsing at the critical point. The reemergence of quasiparticles at temperatures below this scale would then suggest that the interactions responsible for the strongly-correlated non-quasiparticle transport – resulting in linear resistivity but no Wiedemann-Franz law – are only active above this scale. These intermediate low temperatures would seem to be the most interesting, theoretically speaking. It is of interest to see if an extra experimental knob on the system could bring the (disputed) Wiedemann-Franz-recovering scale to zero temperature.

Cuprates

The previous few paragraphs have shown that simple kinematic considerations can organize much of the existing data on the Wiedemann-Franz law in heavy fermions and $Sr_3Ru_2O_7$ in terms of the interplay of the law with a linear in temperature electrical resistivity. If a similar understanding is possible in the cuprates, it offers to shed light on the nature of the linear resistivity in those materials. The existing experimental discussion of Wiedemann-Franz in the cuprates is concerned with very low temperatures in which the electrical resistivity is either constant or increasing as the temperature is lowered, the latter case due to the proximity of localized phases. These results do not probe the linear in temperature resistivity, although they do probe the existence of quasiparticles as $T \to 0$.

The observation of the Wiedemann-Franz law in the overdoped cuprates [3] is consistent with well-defined quasiparticles scattering elastically off impurities at zero temperature. The reported violation of Wiedemann-Franz in underdoped cuprates [10, 11, 12] at very low temperatures (suppressing superconductivity as necessary) can be thought of in three possible ways. Firstly, especially when there is an upturn in the resistivity, violation has been ascribed to localization physics, discussed in the Fermi liquid case towards the end of section 3 above. Localization physics is outside of our conserved-quantity approach to transport, as it comes with the destruction of a coherent Drude peak. We have not set up the tools necessary to describe localization of a non-Fermi liquid. Secondly, a way in which a quasiparticle description would be consistent with a violation of Wiedemann-Franz at $T = 0$, in particular for the case in which $L/L_0 > 1$ is observed with a temperature-independent electrical resistivity [10], is the presence of neutral 'spinon' Fermi surfaces. These would contribute to heat but not electronic conduction. Thirdly, if there are no quasiparticles and strong interactions are present all the way down to zero temperature, then we should not expect Wiedemann-Franz to hold. In this case there is no reason to prefer L/L_0 larger

[10, 11] or smaller [12] than unity as, as we have repeatedly stressed throughout, L_0 is not a legitimate reference point in strongly interacting circumstances that are governed by different kinematics than those underlying the Wiedemann-Franz law. If localization and neutral Fermi surface physics can be ruled out as the explanation for Wiedemann-Franz violation as $T \to 0$ in the cuprates, at least in certain regimes, this would constitute strong evidence for non-quasiparticle transport of the sort described in the earlier sections of this paper. Measurements of L/L_0 over a wider range of temperatures would elucidate this question.

7 Discussion: The two faces of quantum criticality

In this work we have focussed on the kinematics of non-Fermi liquid transport, in contrast to microscopic descriptions specific to any given model. We have discussed simple relationships between quantities of interest. A well-defined framework has been developed within which detailed microscopic computations will be able to tie our considerations to specific theories or materials. The essential point we have attempted to make is that the starting point for any discussion of transport in a metal, in particular if strong interactions are involved, needs to be: What are the almost-conserved quantities governing the low energy dynamics? The answer to this question determines the nature of subsequent investigations, the usefulness or not of Boltzmann equations and the memory matrix, and the role of quantum critical excitations.

The focus on almost-conserved quantities immediately split our discussion of non-Fermi liquid transport into two cases. In the first case, which we might refer to as 'total quantum criticality', all quasiparticles are destroyed by strong interactions at low energies. Therefore all transport processes occur via strongly interacting modes and the only almost conserved quantities are momenta. In the second case, which we might refer to as 'backseat quantum criticality', only a subset of the degrees of freedom are strongly interacting at low energies. The long lived degrees of freedom can acquire unconventional lifetimes due to scattering off the critical modes, but the critical modes themselves do not participate directly in transport as they e.g. relax their momentum too quickly.

In a 'total quantum criticality' scenario the Wiedemann-Franz law is completely off the map. Because the kinematics of almost-conserved quantities is not related to that of a Fermi liquid, one cannot understand these cases by starting with a Fermi liquid and then imagining adding perturbatively additional neutral heat carriers (to increase the thermal conductivity)

or additional inelastic scattering (to increase thermal resistivity). While this intuitively reasonable opposition is relevant for cases where a quasiparticle description of transport is at least partially possible, and has informed recent discussion of the Wiedemann-Franz law in non-Fermi liquids, e.g. [65, 7], it is simply not applicable to strongly interacting non-Fermi liquids.

In our discussion of totally quantum critical transport it was interesting to distinguish the theoretically natural ratio $\overline{\kappa}/\sigma T$, involving the thermal conductivity at vanishing electric field $\overline{\kappa}$, from the usual $\kappa/\sigma T$. For long-lived fermionic quasiparticles $\kappa = \overline{\kappa} \gg \alpha^2 T/\sigma$, for hydrodynamic non-Fermi liquids $\kappa \ll \overline{\kappa}$, while for non-Fermi liquids with patchwise conserved momenta $\kappa \sim \overline{\kappa} \sim \alpha^2 T/\sigma$. The ratio $\overline{\kappa}/\sigma T$ in totally critical non-Fermi liquids was universally related to a ratio of thermodynamic susceptibilities in (15) or, slightly less powerfully, (30). Computation of these susceptibilities in a strongly interacting non-Fermi liquid is likely to be difficult, but perhaps the required susceptibilities can be independently extracted experimentally. We noted in section 6 that the ratio of conductivities does show a fairly regular temperature dependence in the relevant temperature regimes of the candidate totally quantum critical materials $YbRh_2Si_2$ and c-axis $CeCoIn_5$. Measurement of L/L_0 to higher temperatures in these materials and in the underdoped cuprates could corrobate the existence of non-quasiparticle based transport in these materials and search for possibly universal scaling behavior of the ratio of conductivities as a function of temperature.

We have argued that 'backseat quantum criticality' is compatible with the Wiedemann-Franz law if certain circumstances hold. In particular we emphasized that a linear in temperature resistivity can coexist with the Wiedemann-Franz law if it is caused by scattering off a 'generalized phonon' mode above its effective 'Debye temperature'. Crucially, in these cases, Wiedemann-Franz is expected to hold *above* a certain low temperature, as well as at the lowest temperatures, while not holding at intermediate temperatures. It is rather satisfying, as we discussed in section 6, that precisely this behavior has been observed in $CeRhIn_5$ and $Sr_3Ru_2O_7$ away from the critical magnetic fields and in a-axis transport in $CeCoIn_5$ at the critical field. An immediate open question here is to develop a satisfactory microscopic model for the generalized phonon in these materials. To our knowledge there is no microscopic theory of generalized phonons for e.g. spin density wave, metamagnetic or superconductor-metal transitions that might be applicable to these materials. The required theory will presumably be based upon the scattering of cold quasiparticles by hot excitations, in the spirit of [37]. The linear in temperature resistivity observed in these materials would seem therefore to be of a fundamentally different nature to that in the

materials mentioned in the previous paragraph, where the linear resistivity did not coexist with the Wiedemann-Franz law. It is an important lesson, then, that the Wiedemann-Franz law is an interesting diagnostic of quantum critical physics not just at the lowest possible temperature scales, but also higher (low) temperatures.

For $Sr_3Ru_2O_7$ tuned to its critical magnetic field, we faced a confusion. Currently existing data show that while a linear in temperature resistivity persists down to the lowest temperatures prior to ordering, the transport stops obeying the Wiedemann-Franz law at a temperature in the middle of this regime. The simplest scenario compatible with the measurements away from the critical field, from our perspective, is that Wiedemann-Franz be obeyed over the entire temperature range showing a linear in temperature resistivity. Current measurements of the thermal conductivity in this material were performed in less pure samples than those used to detect the linear electrical resistivity. Revisiting these experiments with purer samples seems worthwhile.

Our results in this paper have hinged on the presence of almost-conserved quantities controlling transport. A different set of ideas may be necessary to capture the transport of bad metals. The failure of such metals to exhibit a sharp Drude peak [33] likely indicates the absence of any almost-conserved quantities that overlap with the electrical current.

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References

[1] J. M. Ziman, *Electrons and phonons*, OUP, 1960.

[2] S. Kambe, H. Suderow, T. Fukuhara, J. Flouquet and T. Takimoto, "Spin-Fluctuation Mediated Thermal Conductivity Around the Magnetic Instability of $CeNi_2Ge_2$," J. Low Temp. Phys. **117**, 101 (1999).

[3] C. Proust, E. Boaknin, R. W. Hill, L. Taillefer and A. P. Mackenzie, "Heat Transport in a Strongly Overdoped Cuprate: Fermi Liquid and a Pure d-Wave BCS Superconductor," Phys. Rev. Lett. **89**, 147003 (2002) [arXiv:cond-mat/0202101 [cond-mat.supr-con]].

[4] J. Paglione, M. A. Tanatar, D. G. Hawthorn, F. Ronning, R. W. Hill, M. Sutherland, L. Taillefer and C. Petrovic, "Nonvanishing Energy Scales at the Quantum Critical Point of $CeCoIn_5$," Phys. Rev. Lett. **97**, 106606 (2006) [arXiv:cond-mat/0605124 [cond-mat.str-el]].

[5] R. P. Smith, M. Sutherland, G. G. Lonzarich, S. S. Saxena, N. Kimura, S. Takashima, M. Nohara and H. Takagi, "Marginal breakdown of the Fermi-liquid state on the border of metallic ferromagnetism," Nature **455**, 1220 (2008).

[6] F. Ronning, R. W. Hill, M. Sutherland, D. G. Hawthorn, M. A. Tanatar, J. Paglione, L. Taillefer, M. J. Graf, R. S. Perry, Y. Maeno and A. P. Mackenzie, "Thermal Conductivity in the Vicinity of the Quantum Critical End Point in $Sr_3Ru_2O_7$," Phys. Rev. Lett. **97**, 067005 (2006) [arXiv:cond-mat/0602469 [cond-mat.str-el]].

[7] H. Pfau, S. Hartmann, U. Stockert, P. Sun, S. Lausberg, M. Brando, S. Friedemann, C. Krellner, C. Geibel, S. Wirth, S. Kirchner, E. Abrahams, Q. Si and F. Steglich, "Thermal and electrical transport across a magnetic quantum critical point," Nature **484**, 493 (2012).

[8] Y. Machida, K. Tomokuni, K. Izawa, G. Lapertot, G. Knebel, J.-P. Brison and J. Flouquet "Verification of the Wiedemann-Franz law in $YbRh_2Si_2$ at a quantum critical point," [arXiv:1210.0350 [cond-mat.str-el]].

[9] J-Ph. Reid, M. A. Tanatar, R. Daou, C. Petrovic, L. Taillefer, *in preparation.*

[10] C. Proust, K. Behnia, R. Bel, D. Maude and S. I. Vedeneev, "Heat transport in $Bi_{2+x}Sr_{2-x}CuO_{6+\delta}$: Departure from the Wiedemann-Franz law in the vicinity of the metal-insulator transition," Phys Rev. **B72**, 214511 (2005).

[11] N. Doiron-Leyraud, M. Sutherland, S. Y. Li, L. Taillefer, R. Liang, D. A. Bonn and W. N. Hardy, "Onset of a Boson Mode at the Superconducting Critical Point of Underdoped $YBa_2Cu_3O_y$," Phys. Rev. Lett. **97**, 207001 (2006).

[12] X. F. Sun, B. Lin, X. Zhao, L. Li, S. Komiya, I. Tsukada and Y. Ando, "Deviation from the Wiedemann-Franz law induced by nonmagnetic impurities in overdoped $La_{2-x}Sr_xCuO_4$," Phys. Rev. **B80**, 104510 (2009).

[13] M. A. Tanatar, J. Paglione, C. Petrovic, L. Taillefer, "Anisotropic Violation of the Wiedemann-Franz Law at a Quantum Critical Point," Science **316**, 1320 (2007).

[14] H. K. Pal, V. I. Yudson and D. L. Maslov, "Resistivity of non-Galilean-invariant Fermi- and non-Fermi liquids," Lith. J. Phys. **52**, 142 (2012) [arXiv:1204.3591 [cond-mat.str-el]].

[15] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions*, W. A. Benjamin, Advanced Book Classics, 1975.

[16] A. Rosch and N. Andrei, "Conductivity of a clean one-dimensional wire," Phys. Rev. Lett **85**, 1092 (2000) [arXiv:cond-mat/0002306 [cond-mat.str-el]].

[17] E. Shimshoni, N. Andrei and A. Rosch, "Thermal Conductivity of Spin-1/2 Chains," Phys. Rev. **B68**, 104401 (2003) [arXiv:cond-mat/0304641 [cond-mat.str-el]].

[18] S. A. Hartnoll, P. K. Kovtun, M. Muller and S. Sachdev, "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes," Phys. Rev. **B76**, 144502 (2007) [arXiv:0706.3215 [cond-mat.str-el]].

[19] S. A. Hartnoll and C. P. Herzog, "Impure AdS/CFT correspondence," Phys. Rev. **D77**, 106009 (2008) [arXiv:0801.1693 [hep-th]].

[20] S. A. Hartnoll and D. M. Hofman, "Locally Critical Resistivities from Umklapp Scattering," Phys. Rev. Lett. **108**, 241601 (2012) [arXiv:1201.3917 [hep-th]].

[21] W. Götze and P. Wölfle, "Homogeneous dynamical conductivity of simple metals," Phys. Rev. **B6**, 1226 (1972).

[22] A. Luther, I. Peschel, "Fluctuation Conductivity and Lattice Stability in One Dimension," Phys. Rev. Lett. **32**, 992 (1974).

[23] T. Giamarchi, "Umklapp process and resistivity in one-dimensional fermion systems," Phys. Rev. **B44**, 2905 (1991).

[24] J. Polchinski, "Effective field theory and the Fermi surface," In *Boulder 1992, Proceedings, Recent directions in particle theory* 235-274 [hep-th/9210046].

[25] R. Shankar, "Renormalization group approach to interacting fermions," Rev. Mod. Phys. **66**, 129 (1994).

[26] G. .V. Chester and A. Thellung, "The law of Wiedemann and Franz," Proc. Phys. Soc. **77**, 1005 (1961).

[27] G. Catelani and I. L. Aleiner, "Interaction corrections to thermal transport coefficients in disordered metals: the quantum kinetic equation approach," JETP, **100**, 331 (2005) [arXiv:cond-mat/0405333 [cond-mat.dis-nn]].

[28] K. Michaeli and A. M. Finkel'stein, "Quantum kinetic approach for studying thermal transport in the presence of electron-electron interactions and disorder," Phys. Rev. **B80**, 115111 (2009) [arXiv:0909.2482 [cond-mat.dis-nn]].

[29] F. D. M. Haldane, "Luttinger's Theorem and Bosonization of the Fermi Surface," in *Perspectives in Many-Particle Physics* eds. R. A. Broglia and J. R. Schrieffer (North-Holland, Amsterdam 1994) [arXiv:cond-mat/0505529 [cond-mat.str-el]].

[30] A. Houghton and J. B. Marston, "Bosonization and fermion liquids in dimensions greater than one," Phys. Rev. **B48**, 7790 (1993) [arXiv:cond-mat/9210007].

[31] A. H. Castro Neto and E. Fradkin, "Bosonization of the low energy excitations of Fermi liquids," Phys. Rev. Lett. **72**, 1393 (1994) [arXiv:cond-mat/9304014].

[32] V. J. Emery and S. A. Kivelson, "Superconductivity in Bad Metals," Phys. Rev. Lett. **74**, 3253 (1995).

[33] N. E. Hussey, K. Takenaka and H. Takagi, "Universality of the Mott-Ioffe-Regel limit in metals," Phil. Mag. **84**, 2847 (2004) [cond-mat/0404263].

[34] M. A. Metlitski and S. Sachdev, "Quantum phase transitions of metals in two spatial dimensions: II. Spin density wave order," Phys. Rev. B **82**, 075128 (2010) [arXiv:1005.1288 [cond-mat.str-el]].

[35] T. Senthil, "Theory of a continuous Mott transition in two dimensions," Phys. Rev. **B78**, 045109 (2008) [arXiv:0804.1555 [cond-mat.str-el]].

[36] W. Witczak-Krempa, P. Ghaemi, T. Senthil, Y. B. Kim "Universal transport near a quantum critical Mott transition in two dimensions," Phys. Rev. **B86**, 245102 (2012) [arXiv:1206.3309 [cond-mat.str-el]].

[37] S. A. Hartnoll, D. M. Hofman, M. A. Metlitski and S. Sachdev, "Quantum critical response at the onset of spin density wave order in two-dimensional metals," Phys. Rev. B **84**, 125115 (2011) [arXiv:1106.0001 [cond-mat.str-el]].

[38] A. Donos and S. A. Hartnoll, "Metal-insulator transition in holography," arXiv:1212.2998 [hep-th].

[39] B. Spivak and S. A. Kivelson, "Transport in two dimensional electronic micro-emulsions," Ann. Phys. **321**, 2071 (2006) [arXiv:cond-mat/0510422 [cond-mat.mes-hall]].

[40] A. V. Andreev, S. A. Kivelson and B. Spivak "Hydrodynamic description of transport in strongly correlated electron systems," Phys. Rev. Lett. **106**, 256804 (2011) [arXiv:1011.3068 [cond-mat.mes-hall]].

[41] N. Ogawa, T. Takayanagi and T. Ugajin, "Holographic Fermi Surfaces and Entanglement Entropy," JHEP **1201**, 125 (2012) [arXiv:1111.1023 [hep-th]].

[42] L. Huijse, S. Sachdev and B. Swingle, "Hidden Fermi surfaces in compressible states of gauge-gravity duality," Phys. Rev. B **85**, 035121 (2012) [arXiv:1112.0573 [cond-mat.str-el]].

[43] S. A. Hartnoll and E. Shaghoulian, "Spectral weight in holographic scaling geometries," JHEP **1207**, 078 (2012) [arXiv:1203.4236 [hep-th]].

[44] R. J. Anantua, S. A. Hartnoll, V. L. Martin and D. M. Ramirez, "The Pauli exclusion principle at strong coupling: Holographic matter and momentum space," arXiv:1210.1590 [hep-th].

[45] D. T. Son and A. O. Starinets, "Hydrodynamics of r-charged black holes," JHEP **0603**, 052 (2006) [hep-th/0601157].

[46] M. A. Metlitski and S. Sachdev, "Quantum phase transitions of metals in two spatial dimensions: I. Ising-nematic order," Phys. Rev. B **82**, 075127 (2010) [arXiv:1001.1153 [cond-mat.str-el]].

[47] J. Polchinski, "Low-energy dynamics of the spinon gauge system," Nucl. Phys. B **422**, 617 (1994) [cond-mat/9303037].

[48] B. L. Altshuler, L. B. Ioffe and A. J. Millis, "Low-energy properties of fermions with singular interactions," Phys. Rev. B **50**, 14048 (1994).

[49] S.-S. Lee, "Low energy effective theory of Fermi surface coupled with U(1) gauge field in 2+1 dimensions," Phys. Rev. **B 80**, 165102 (2009) [arXiv:0905.4532 [cond-mat.str-el]].

[50] L. Fritz and S. Sachdev, "Signatures of the nematic ordering transitions in the thermal conductivity of d-wave superconductors," Phys. Rev. **B80**, 144503 (2009) [arXiv:0901.3530 [cond-mat.str-el]].

[51] D. Podolsky, A. Vishwanath, J. Moore and S. Sachdev, "Thermoelectric transport near pair breaking quantum phase transition out of d-wave superconductivity," Phys. Rev. **B 75**, 014520 (2007) [arXiv:cond-mat/0510597 [cond-mat.str-el]].

[52] J.-F. Mercure, A. W. Rost, E. C. T. O'Farrell, S. K. Goh, R. S. Perry, M. L. Sutherland, S. A. Grigera, R. A. Borzi, P. Gegenwart, A. S. Gibbs and A. P. Mackenzie, "Quantum oscillations near the metamagnetic transition in $Sr_3Ru_2O_7$," Phys. Rev. **B 81**, 235103 (2010) [arXiv:cond-mat/0403390 [cond-mat.str-el]].

[53] R. Hlubina and T. M. Rice, "Resistivity as a function of temperature for models with hot spots on the Fermi surface," Phys. Rev. **B51**, 9253 (1994).

[54] A. Rosch, "Interplay of Disorder and Spin Fluctuations in the Resistivity near a Quantum Critical Point," Phys. Rev. Lett. **82**, 4280 (1999) [arXiv:cond-mat/9810260].

[55] J. A. N. Bruin, H. Sakai, R. S. Perry, and A. P. Mackenzie, "Similarity of Scattering Rates in Metals Showing T-Linear Resistivity," Science **339**, 804 (2013).

[56] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein. "Phenomenology of the normal state of Cu-O high-temperature superconductors," Phys. Rev. Lett. **63**, 1996 (1989).

[57] G. R. Stewart, "Non-Fermi-liquid behavior in d-and f-electron metals," Rev. Mod. Phys. **73**, 797 (2001).

[58] A. W. Rost, S. A. Grigera, J. A. N. Bruin, R. S. Perry, D. Tian, S. Raghu, S. A. Kivelson, and A. P. Mackenzie, "Thermodynamics of phase formation in the quantum critical metal $Sr_3Ru_2O_7$," Proc. Nat. Acad. Sci. **108**, 16549 (2011) [arXiv:1108.3554 [cond-mat.str-el]].

[59] S. Kimura, T. Nishi, J. Sichelschmidt, V. Voevodin, J. Ferstl, C. Geibel and F. Steglich, "Optical conductivity of a non-Fermi-liquid material $YbRh_2Si_2$," J. Mag. and Mag. Mat. **272**, 36 (2004).

[60] F. P. Mena, D. van der Marel and J. L. Sarrao, "Optical conductivity of $CeMIn_5$ (M=Co, Rh, Ir)," Phys. Rev. **72**, 045119 (2005).

[61] Y. Onose, N. P. Ong and C. Petrovic, "The Lorenz number in $CeCoIn_5$ inferred from the thermal and charge Hall currents," EPL **80** 37005 (2007) [arXiv:0706.2674 [cond-mat.str-el]].

[62] J. Paglione, M. A. Tanatar, D. G. Hawthorn, R. W. Hill, F. Ronning, M. Sutherland, L. Taillefer, C. Petrovic, and P. C. Canfield, "Heat Transport as a Probe of Electron Scattering by Spin Fluctuations: The Case of Antiferromagnetic $CeRhIn_5$," Phys. Rev. Lett. **94**, 216602 (2005) [arXiv:cond-mat/0404269 [cond-mat.str-el]].

[63] P. Gegenwart, J. Custers, C. Geibel, K. Neumaier, T. Tayama, K. Tenya, O. Trovarelli and F. Steglich, "Magnetic-Field Induced Quantum Critical Point in $YbRh_2Si_2$," Phys. Rev. Lett. **89**, 56402 (2002).

[64] J. K. Dong, Y. Tokiwa, S. L. Budko, P. C. Canfield, P. Gegenwart, "Anomalous reduction of the Lorenz ratio at the quantum critical point in YbAgGe," arXiv:1304.2210 [cond-mat.str-el].

[65] K-S. Kim, and C. Pépin, "Violation of the Wiedemann-Franz Law at the Kondo Breakdown Quantum Critical Point," Phys. Rev. Let. **102**, 156404 (2009) [arXiv:0811.0638 [cond-mat.str-el]].